Wydział Mechaniczny Energetyki i Lotnictwa
Zakład Wytrzymałości Materiałów i Konstrukcji

Metoda elementów skończonych (MES1)

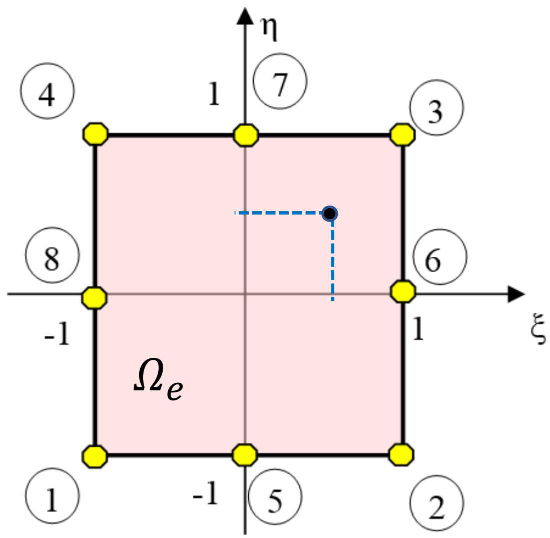
Wykład 4A. 8-węzłowy element czworokątny

03.2024

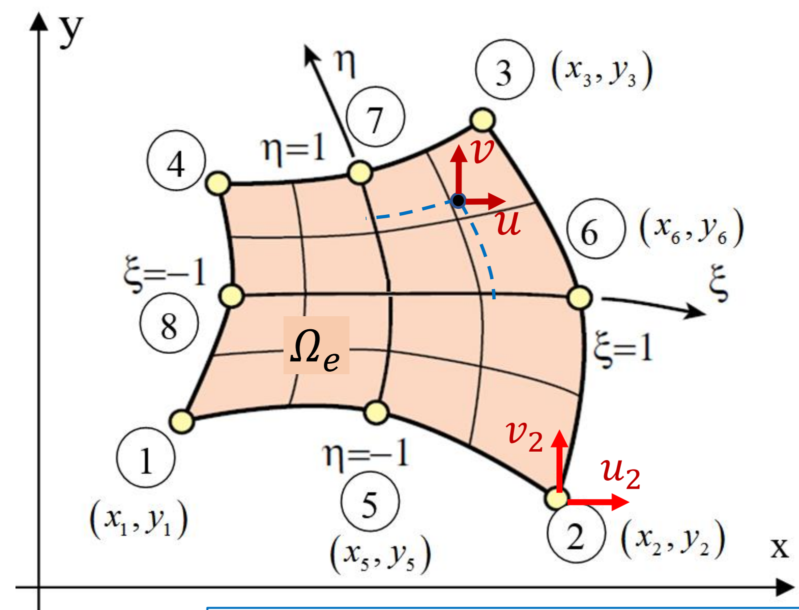
8-węzłowy 2D element czworokątny (dokładność, nieregularne kształty)

Element 8-węzłowy jest zdefiniowany przez osiem węzłów o dwóch stopniach swobody w każdym węźle u_i, v_i. Zapewnia dokładniejsze wyniki i może uwzględniać nieregularne kształty bez znacznej utraty dokładności.

układ współrzędnych naturalnych



układ współrzędnych kartezjańskich



$n = 8 \;\; ; \;\; n_p = 2 \;\; \rightarrow \;\; n_e = n \cdot n_p = 16$

Odwzorowanie geometryczne: $\boxed{(\xi, \eta) \rightarrow (x, y)}$

$$(-1,-1) \rightarrow (x_1, y_1) \quad (1,-1) \rightarrow (x_2, y_2) \quad (1,1) \rightarrow (x_3, y_3) \quad (-1,1) \rightarrow (x_4, y_4)$$

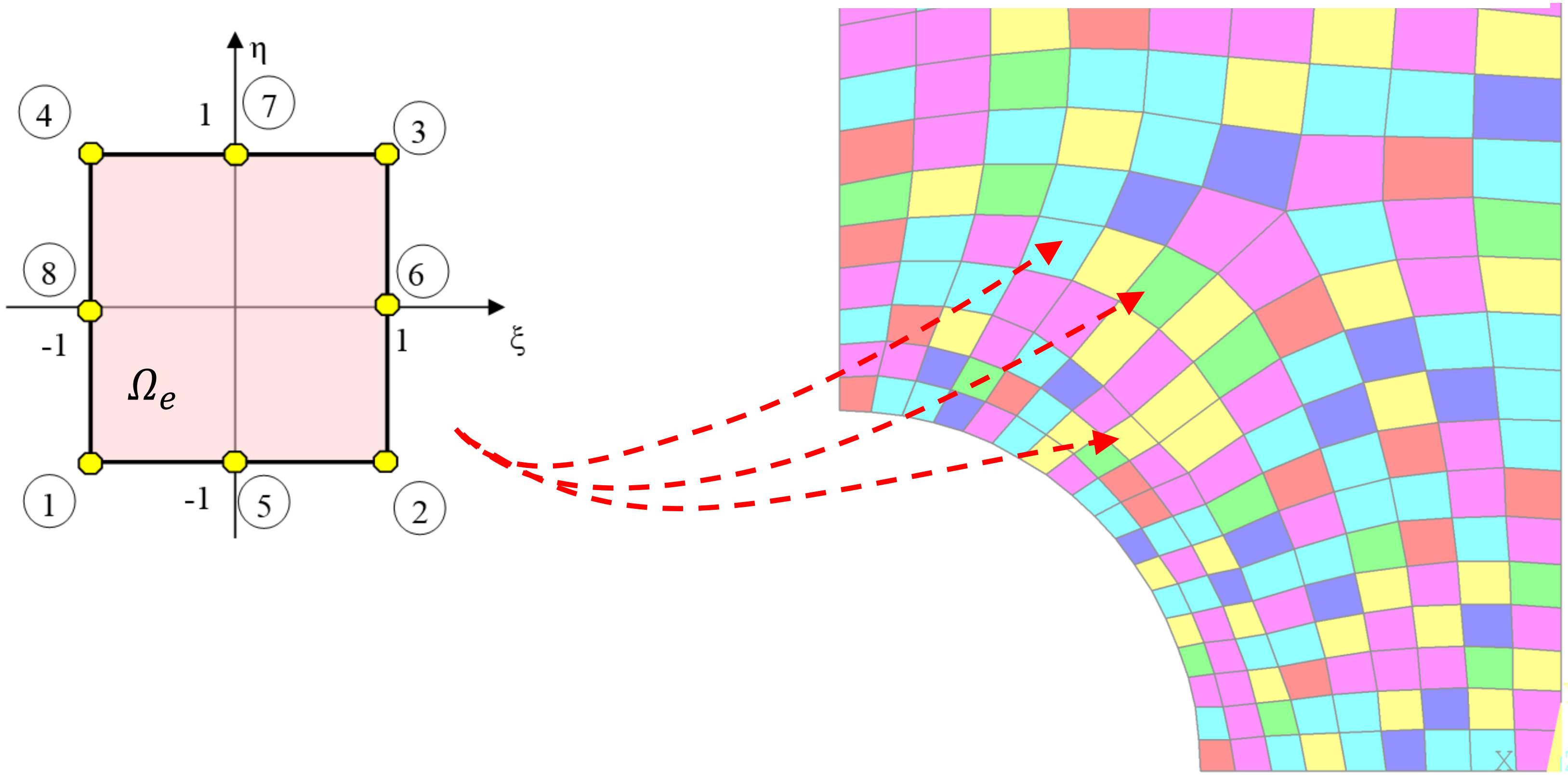
$$(0,-1) \rightarrow (x_5, y_5) \quad (1,0) \rightarrow (x_6, y_6) \quad (0,1) \rightarrow (x_7, y_7) \quad (-1,0) \rightarrow (x_8, y_8)$$

Element odniesienia a elementy rzeczywiste

Jeden element odniesienia odwzorowuje się na każdy element rzeczywisty siatki

element odniesienia

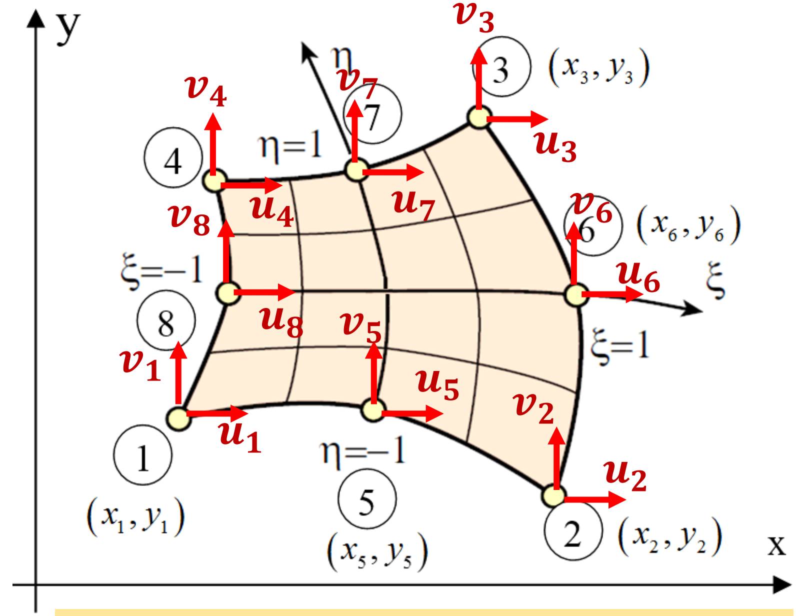
siatka elementów modelu



Odwzorowanie izoparametryczne

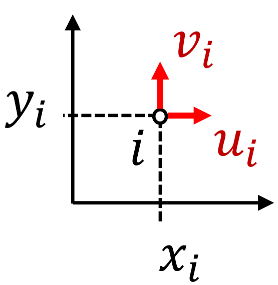
wektory współrzędnych węzłowych

$$\underset{8\times 1}{\{x_i\}_e} = \begin{Bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_8 \end{Bmatrix} \quad ; \quad \underset{8\times 1}{\{y_i\}_e} = \begin{Bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_8 \end{Bmatrix}$$



$n = 8$; $n_p = 2$ $\rightarrow$ $n_e = n \cdot n_p = 16$

wektor lokalny parametrów węzłowych



$$\underset{16\times 1}{\{q\}_e} = \begin{Bmatrix} q_1 \\ q_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ q_{16} \end{Bmatrix}_e = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ u_8 \\ v_8 \end{Bmatrix}_e$$

Odwzorowanie izoparametryczne –te same funkcje użyte są do opisu geometrii i pola przemieszczeń

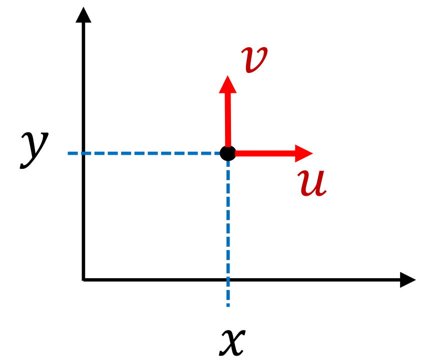
Odwzorowanie izoparametryczne

macierz funkcji kształtu*:*

$$\underset{2\times 16}{[N(\xi,\eta)]} = \begin{bmatrix} N_1(\xi,\eta) & 0 & N_2(\xi,\eta) & 0 & \dots & N_8(\xi,\eta) & 0 \\ 0 & N_1(\xi,\eta) & 0 & N_2(\xi,\eta) & \dots & 0 & N_8(\xi,\eta) \end{bmatrix}$$

wektor funkcji kształtu*:*

$$\underset{1\times 8}{\lfloor N(\xi,\eta)\rfloor} = \lfloor N_1(\xi,\eta), N_2(\xi,\eta), \dots, N_8(\xi,\eta)\rfloor$$



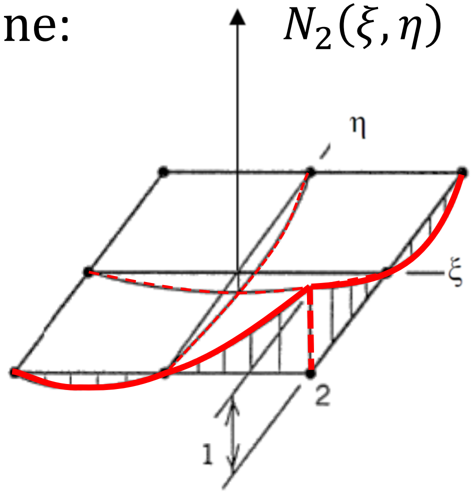
położenie i przemieszczenie dowolnego punktu:

$$\boxed{x = \underset{1\times 8}{\lfloor N(\xi,\eta)\rfloor}\underset{8\times 1}{\{x_i\}_e} \quad ; \quad y = \underset{1\times 8}{\lfloor N(\xi,\eta)\rfloor}\underset{8\times 1}{\{y_i\}_e}}$$

$$\boxed{\underset{2\times 1}{\{u\}} = \begin{Bmatrix} u \\ v \end{Bmatrix} = \underset{2\times 16}{[N(\xi,\eta)]}\underset{16\times 1}{\{q\}_e}}$$

Funkcje kształtu 8-węzłowego elementu czworokątnego

węzły narożne:



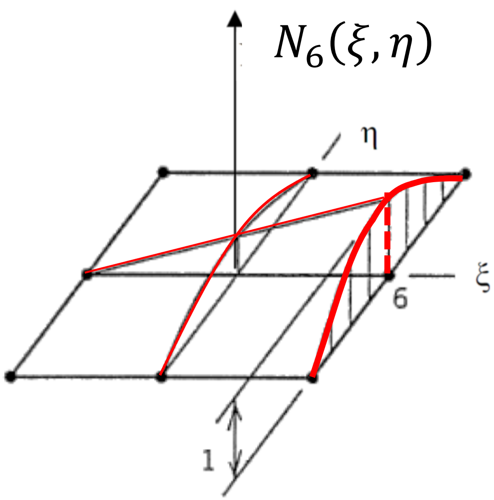
$$N_1(\xi,\eta) = -\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta)$$

$$N_2(\xi,\eta) = -\frac{1}{4}(1+\xi)(1-\eta)(1-\xi+\eta)$$

$$N_3(\xi,\eta) = -\frac{1}{4}(1+\xi)(1+\eta)(1-\xi-\eta)$$

$$N_4(\xi,\eta) = -\frac{1}{4}(1-\xi)(1+\eta)(1+\xi-\eta)$$

węzły w środku boków:



$$N_5(\xi,\eta) = \frac{1}{2}(1-\xi^2)(1-\eta)$$

$$N_6(\xi,\eta) = \frac{1}{2}(1+\xi)(1-\eta^2)$$

$$N_7(\xi,\eta) = \frac{1}{2}(1-\xi^2)(1+\eta)$$

$$N_8(\xi,\eta) = \frac{1}{2}(1-\xi)(1-\eta^2)$$

Transformacja pomiędzy układem naturalnym a kartezjańskim

pochodne cząstkowe dowolnej funkcji współrzędnych (x, y) wzgędem (ξ, η):

$$\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial f}{\partial \eta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \eta}$$

$$\Longrightarrow \quad \begin{Bmatrix} \dfrac{\partial f}{\partial \xi} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix} = \boxed{\begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}} \begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix} = \underset{2\times 2}{[J]} \cdot \begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix}$$

macierz Jacobiego

pochodne cząstkowe dowolnej funkcji współrzędnych (ξ, η) względem (x, y):

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$\Longrightarrow \quad \begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \eta}{\partial x} \\ \dfrac{\partial \xi}{\partial y} & \dfrac{\partial \eta}{\partial y} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial f}{\partial \xi} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix} = \underset{2\times 2}{[J]}^{-1} \begin{Bmatrix} \dfrac{\partial f}{\partial \xi} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix}$$

odwrócona macierz Jacobiego

Transformacja pomiędzy układem naturalnym a kartezjańskim

operatory różniczkowe:

$$\begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} = \boxed{\underset{2\times 2}{[J]} \cdot \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix}} \quad ; \quad \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \eta}{\partial x} \\ \dfrac{\partial \xi}{\partial y} & \dfrac{\partial \eta}{\partial y} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix} = \boxed{\underset{2\times 2}{[J]^{-1}} \begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix}}$$

macierz Jacobiego *odwrócona macierz Jacobiego*

operatory różniczkowe:

$$\begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix} = \boxed{\underset{2\times 2}{[J]} \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix}} = \underset{2\times 2}{[J]} \cdot \boxed{\underset{2\times 2}{[J]^{-1}} \begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix}} = \underset{2\times 2}{[I]} \cdot \begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix}$$

macierz jednostkowa

Jak znaleźć macierz odwrotną Jacobiego?

$$[J]=\begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}$$

odwrócona macierz Jacobiego:

$$\underset{2\times 2}{[J]^{-1}} = \frac{1}{det[J]}\left(\underset{2\times 2}{[J]^{C}}\right)^{T} = \frac{1}{det[J]}\left(\begin{bmatrix} \dfrac{\partial y}{\partial \eta} & -\dfrac{\partial x}{\partial \eta} \\ -\dfrac{\partial y}{\partial \xi} & \dfrac{\partial x}{\partial \xi} \end{bmatrix}\right)^{T} = \frac{1}{det[J]}\begin{bmatrix} \dfrac{\partial y}{\partial \eta} & -\dfrac{\partial y}{\partial \xi} \\ -\dfrac{\partial x}{\partial \eta} & \dfrac{\partial x}{\partial \xi} \end{bmatrix} =$$

macierz dopełnień algebraicznych

$$= \begin{bmatrix} \dfrac{1}{det[J]}\dfrac{\partial y}{\partial \eta} & -\dfrac{1}{det[J]}\dfrac{\partial y}{\partial \xi} \\ -\dfrac{1}{det[J]}\dfrac{\partial x}{\partial \eta} & \dfrac{1}{det[J]}\dfrac{\partial x}{\partial \xi} \end{bmatrix}$$

współrzędne węzłowe

aproksymacja geometrii

$$\underset{2\times 2}{[J]^{-1}} = \begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \eta}{\partial x} \\ \dfrac{\partial \xi}{\partial y} & \dfrac{\partial \eta}{\partial y} \end{bmatrix} \rightarrow$$

$$\frac{\partial \xi}{\partial x} = \frac{1}{det[J]}\frac{\partial y}{\partial \eta} = \frac{1}{det[J]}\frac{\partial(\lfloor N(\xi,\eta)\rfloor\{y_i\}_e)}{\partial \eta} = \frac{1}{det[J]}\underset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \eta}}\underset{8\times 1}{\{y_i\}_e}$$

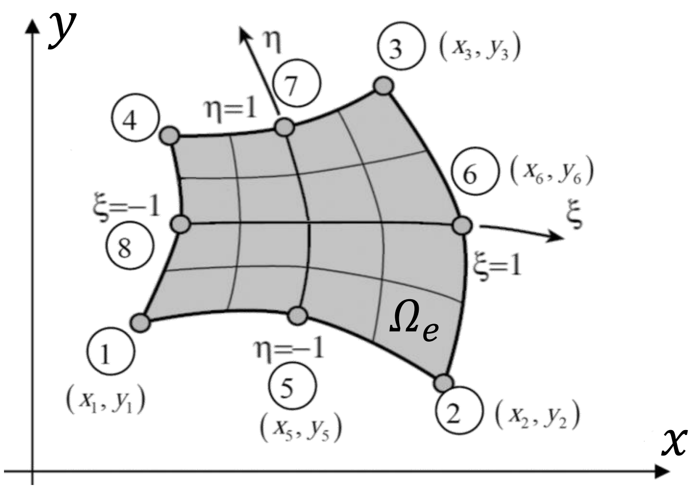
$$\frac{\partial \eta}{\partial x} = -\frac{1}{det[J]}\frac{\partial y}{\partial \xi} = -\frac{1}{det[J]}\frac{\partial(\lfloor N(\xi,\eta)\rfloor\{y_i\}_e)}{\partial \xi} = -\frac{1}{det[J]}\underset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \xi}}\underset{8\times 1}{\{y_i\}_e}$$

$$\frac{\partial \xi}{\partial y} = -\frac{1}{det[J]}\frac{\partial x}{\partial \eta} = -\frac{1}{det[J]}\frac{\partial(\lfloor N(\xi,\eta)\rfloor\{x_i\}_e)}{\partial \eta} = -\frac{1}{det[J]}\underset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \eta}}\underset{8\times 1}{\{x_i\}_e}$$

$$\frac{\partial \eta}{\partial y} = \frac{1}{det[J]}\frac{\partial x}{\partial \xi} = \frac{1}{det[J]}\frac{\partial(\lfloor N(\xi,\eta)\rfloor\{x_i\}_e)}{\partial \xi} = \frac{1}{det[J]}\underset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \xi}}\underset{8\times 1}{\{x_i\}_e}$$

Jak wyliczyć wyznacznik macierzy Jacobiego?

wyznacznik macierzy Jacobiego:



aproksymacja geometrii

$$det[J] = det\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta} =$$

$$= \frac{\partial(\lfloor N(\xi,\eta)\rfloor\{x_i\}_e)}{\partial \xi}\frac{\partial(\lfloor N(\xi,\eta)\rfloor\{y_i\}_e)}{\partial \eta} - \frac{\partial(\lfloor N(\xi,\eta)\rfloor\{y_i\}_e)}{\partial \xi}\frac{\partial(\lfloor N(\xi,\eta)\rfloor\{x_i\}_e)}{\partial \eta} =$$

$$= \left(\underset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \xi}}\underset{8\times 1}{\{x_i\}_e}\right)\left(\underset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \eta}}\underset{8\times 1}{\{y_i\}_e}\right) - \left(\underset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \xi}}\underset{8\times 1}{\{y_i\}_e}\right)\left(\underset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \eta}}\underset{8\times 1}{\{x_i\}_e}\right)$$

(można wyliczyć w dowolnym punkcie domeny Ω_e)

Wyliczenie macierzy gradientu

operatory różniczkowe w układzie współrzędnych (x, y):

$$\begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} = \underset{2\times 2}{[J]^{-1}} \begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{1}{det[J]}\dfrac{\partial y}{\partial \eta} & -\dfrac{1}{det[J]}\dfrac{\partial y}{\partial \xi} \\ -\dfrac{1}{det[J]}\dfrac{\partial x}{\partial \eta} & \dfrac{1}{det[J]}\dfrac{\partial x}{\partial \xi} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix} \rightarrow$$

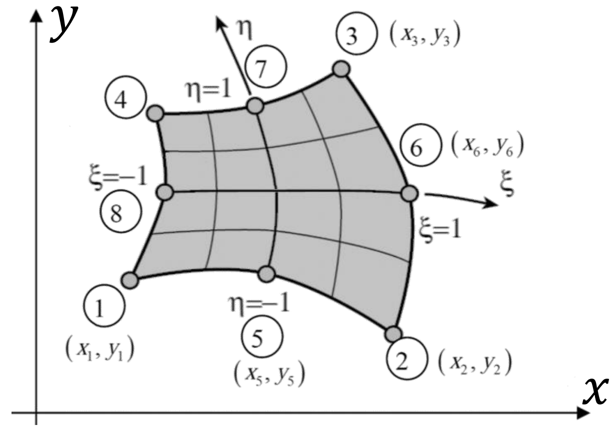
$$\frac{\partial}{\partial x} = \frac{1}{det[J]}\frac{\partial y}{\partial \eta}\frac{\partial}{\partial \xi} - \frac{1}{det[J]}\frac{\partial y}{\partial \xi}\frac{\partial}{\partial \eta} \quad ; \quad \frac{\partial}{\partial y} = -\frac{1}{det[J]}\frac{\partial x}{\partial \eta}\frac{\partial}{\partial \xi} + \frac{1}{det[J]}\frac{\partial x}{\partial \xi}\frac{\partial}{\partial \eta}$$

Macierz gradientu dla warunku PSN lub PSO:

$$\underset{3\times 2}{[R]} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} = \begin{bmatrix} \left(\dfrac{1}{det[J]}\dfrac{\partial y}{\partial \eta}\dfrac{\partial}{\partial \xi} - \dfrac{1}{det[J]}\dfrac{\partial y}{\partial \xi}\dfrac{\partial}{\partial \eta}\right) & 0 \\ 0 & \left(\dfrac{1}{det[J]}\dfrac{\partial x}{\partial \xi}\dfrac{\partial}{\partial \eta} - \dfrac{1}{det[J]}\dfrac{\partial x}{\partial \eta}\dfrac{\partial}{\partial \xi}\right) \\ \left(\dfrac{1}{det[J]}\dfrac{\partial x}{\partial \xi}\dfrac{\partial}{\partial \eta} - \dfrac{1}{det[J]}\dfrac{\partial x}{\partial \eta}\dfrac{\partial}{\partial \xi}\right) & \left(\dfrac{1}{det[J]}\dfrac{\partial y}{\partial \eta}\dfrac{\partial}{\partial \xi} - \dfrac{1}{det[J]}\dfrac{\partial y}{\partial \xi}\dfrac{\partial}{\partial \eta}\right) \end{bmatrix} = \underset{3\times 2}{[R(\xi, \eta)]}$$

Macierz gradientu dla warunku PSN lub PSO

$$\underset{3\times 2}{[R(\xi,\eta)]} = \begin{bmatrix} \dfrac{\partial}{\partial x}(\xi,\eta) & 0 \\ 0 & \dfrac{\partial}{\partial y}(\xi,\eta) \\ \dfrac{\partial}{\partial y}(\xi,\eta) & \dfrac{\partial}{\partial x}(\xi,\eta) \end{bmatrix}$$



$$\underset{8\times 1}{\{x_i\}_e} = \begin{Bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_8 \end{Bmatrix}$$

$$\underset{8\times 1}{\{y_i\}_e} = \begin{Bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_8 \end{Bmatrix}$$

$$\frac{\partial}{\partial x}(\xi,\eta) = \frac{\left(\overset{1\times 8}{\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\eta}}\ \overset{8\times 1}{\{y_i\}_e}\right)\dfrac{\partial}{\partial\xi} - \left(\overset{1\times 8}{\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\xi}}\ \overset{8\times 1}{\{y_i\}_e}\right)\dfrac{\partial}{\partial\eta}}{\left(\underset{1\times 8}{\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\xi}}\ \underset{8\times 1}{\{x_i\}_e}\right)\left(\underset{1\times 8}{\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\eta}}\ \underset{8\times 1}{\{y_i\}_e}\right) - \left(\underset{1\times 8}{\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\xi}}\ \underset{8\times 1}{\{y_i\}_e}\right)\left(\underset{1\times 8}{\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\eta}}\ \underset{8\times 1}{\{x_i\}_e}\right)}$$

$$\frac{\partial}{\partial y}(\xi,\eta) = \frac{\left(\overset{1\times 8}{\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\xi}}\ \overset{8\times 1}{\{x_i\}_e}\right)\dfrac{\partial}{\partial\eta} - \left(\overset{1\times 8}{\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\eta}}\ \overset{8\times 1}{\{x_i\}_e}\right)\dfrac{\partial}{\partial\xi}}{\left(\underset{1\times 8}{\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\xi}}\ \underset{8\times 1}{\{x_i\}_e}\right)\left(\underset{1\times 8}{\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\eta}}\ \underset{8\times 1}{\{y_i\}_e}\right) - \left(\underset{1\times 8}{\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\xi}}\ \underset{8\times 1}{\{y_i\}_e}\right)\left(\underset{1\times 8}{\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\eta}}\ \underset{8\times 1}{\{x_i\}_e}\right)}$$

Energia odkształcenia 8-węzłowego elementu czworokątnego

wektor odkształcenia dla warunku PSN lub PSO:

$$\underset{3\times 1}{\{\varepsilon\}} = \underset{3\times 2}{[R(\xi,\eta)]}\underset{2\times 1}{\{u\}} = \underset{3\times 2}{[R(\xi,\eta)]}\underset{2\times 16}{[N(\xi,\eta)]}\underset{16\times 1}{\{q\}_e} = \underset{3\times 16}{[B(\xi,\eta)]}\underset{16\times 1}{\{q\}_e}$$

energia sprężysta w elemencie skończonym:

$$U_e = \frac{1}{2}\int_{\Omega_e}\underset{1\times 3}{\lfloor\varepsilon\rfloor}\underset{3\times 1}{\{\sigma\}}\, d\Omega_e = \frac{1}{2}\underset{1\times 16}{\lfloor q\rfloor_e}\left(t_e\int_{A_e}\underset{16\times 3}{[B(\xi,\eta)]^T}\underset{3\times 3}{[D]}\underset{3\times 16}{[B(\xi,\eta)]}\, dxdy\right)\underset{16\times 1}{\{q\}_e} =$$

$$\underset{3\times 1}{\{\sigma\}} = \underset{3\times 3}{[D]}\ \underset{3\times 1}{\{\varepsilon\}}$$

$$= \frac{1}{2}\lfloor q\rfloor_e[k]_e\{q\}_e$$

$$\underset{1\times 3}{\lfloor\varepsilon\rfloor} = \underset{1\times 16}{\lfloor q\rfloor_e}\underset{16\times 3}{[B(\xi,\eta)]^T} \qquad \underset{3\times 1}{\{\varepsilon\}} = \underset{3\times 16}{[B(\xi,\eta)]}\underset{16\times 1}{\{q\}_e}$$

$$\left(\int_{A_e} f(x,y)\, dxdy = \int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)det[J]\, d\xi d\eta\right)$$

lokalna macierz sztywności:

$$\boxed{\underset{16\times 16}{[k]_e} = t_e\int_{-1}^{1}\int_{-1}^{1}\underset{16\times 3}{[B(\xi,\eta)]^T}\underset{3\times 3}{[D]}\underset{3\times 16}{[B(\xi,\eta)]}det[J]\, d\xi d\eta}$$

(wyliczana numerycznie)

Energia potencjalna obciążenia i wektor obciążenia zastępczego

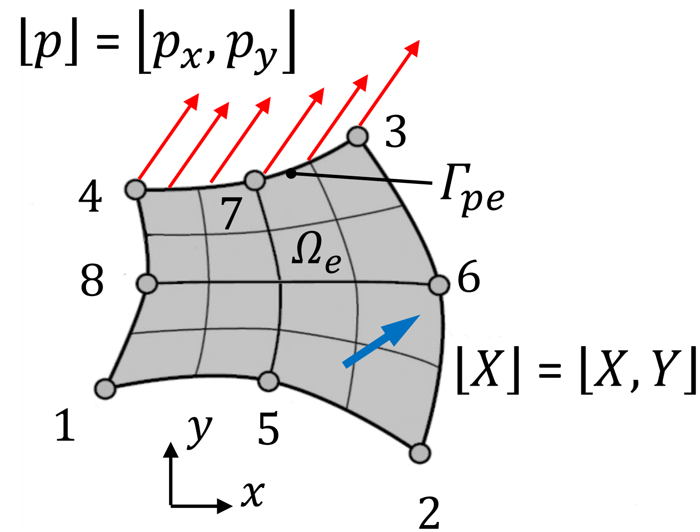
energia potencjalna obciążeniain w elemencie:

$$W_e = \int\limits_{\Omega_e} \underset{1\times 2}{\lfloor X \rfloor} \underset{2\times 1}{\{u\}}\, d\Omega_e + \int\limits_{\Gamma_{pe}} \underset{1\times 2}{\lfloor p \rfloor} \underset{2\times 1}{\{u\}}\, d\Gamma_{pe} =$$

$$\underset{2\times 1}{\{u\}} = \underset{2\times 16}{[N]} \underset{16\times 1}{\{q\}_e}$$

$$= \Big(\int\limits_{\Omega_e} \underset{1\times 2}{\lfloor X \rfloor} \underset{2\times 16}{[N]}\, d\Omega_e + \int\limits_{\Gamma_{pe}} \underset{1\times 2}{\lfloor p \rfloor} \underset{2\times 16}{[N]}\, d\Gamma_{pe}\Big) \underset{16\times 1}{\{q\}_e} =$$

$$= \big(\underset{1\times 16}{\lfloor F^X \rfloor_e} + \underset{1\times 16}{\lfloor F^p \rfloor_e}\big) \underset{16\times 1}{\{q\}_e} = \underset{1\times 16}{\lfloor F \rfloor_e} \underset{16\times 1}{\{q\}_e}$$

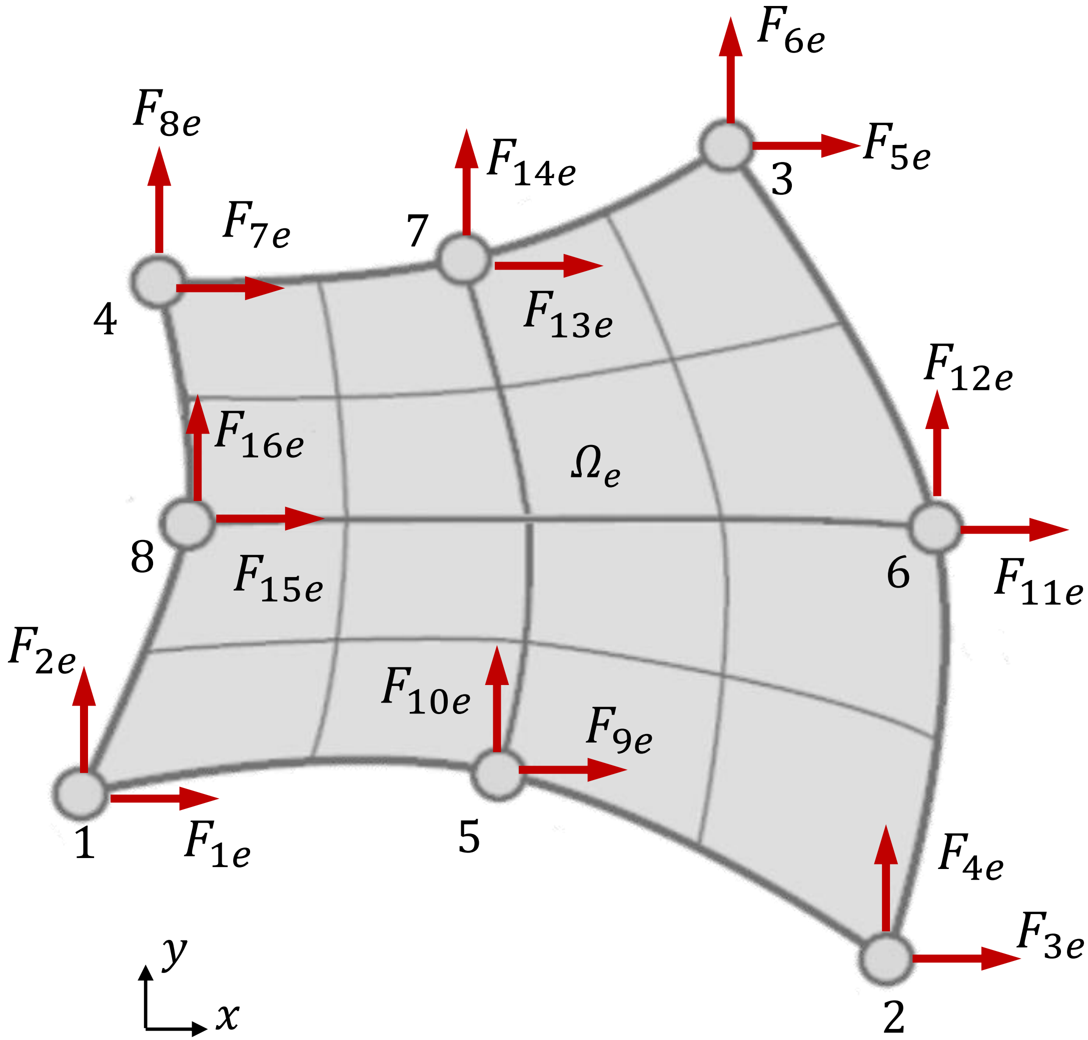


$$\underset{1\times 16}{\lfloor F^X \rfloor_e} = t_e \int_{-1}^{1}\int_{-1}^{1} \underset{1\times 2}{\lfloor X(\xi,\eta) \rfloor} \underset{2\times 16}{[N(\xi,\eta)]}\, det[J]\, d\xi d\eta$$

$$\underset{1\times 16}{\lfloor F^p \rfloor_e} = \int\limits_{\Gamma_{pe}} \underset{1\times 2}{\lfloor p \rfloor} \underset{2\times 16}{[N]} d\Gamma_{pe}$$

Wektor obciążenia zastępczego w elemencie 8-węzłowym

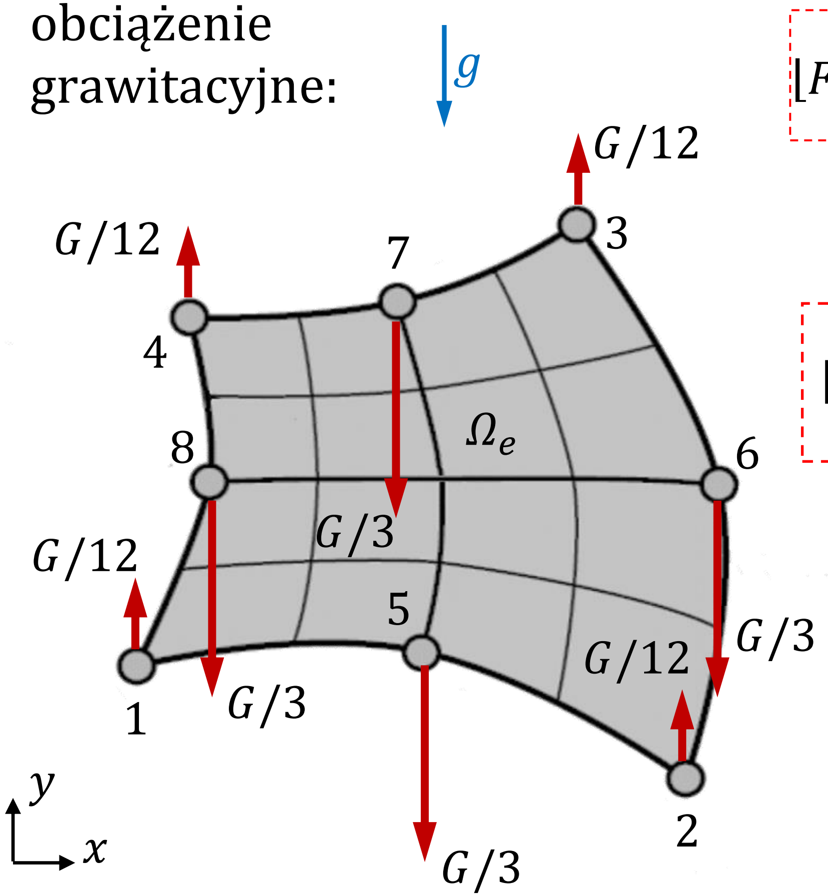
$\underset{16\times 1}{\lfloor F \rfloor_e}$



$$\{F\}_e = \begin{Bmatrix} F_1 \\ F_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ F_{16} \end{Bmatrix}_e$$

Przykład. Wektor obciążenia zastępczego od sił masowych *(grawitacji)*

obciążenie grawitacyjne:



$$\lfloor F^X \rfloor_e = t_e \int_{-1}^{1}\int_{-1}^{1} \lfloor X(\xi,\eta) \rfloor [N(\xi,\eta)]\; det[J]\; d\xi d\eta$$

$$\underset{1\times 2}{\lfloor X \rfloor}$$

$$\underset{1\times 16}{\lfloor F^X \rfloor_e} = t_e \int_{-1}^{1}\int_{-1}^{1} \lfloor 0, -\rho g \rfloor \underset{2\times 16}{[N(\xi,\eta)]}\; det[J]\; d\xi d\eta$$

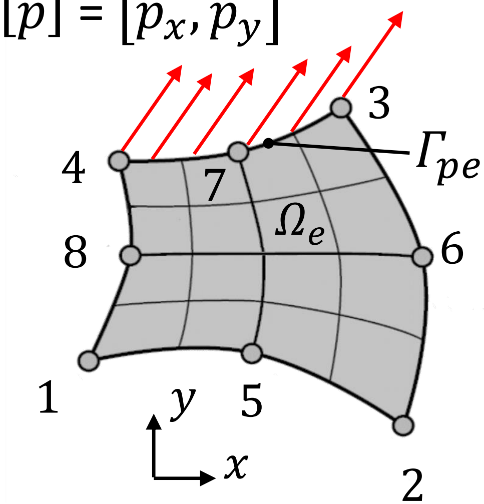
$$G = \rho g \Omega_e \; (N)$$

$$\underset{1\times 16}{\lfloor F \rfloor_e} = \left\lfloor 0, \frac{G}{12}, 0, \frac{G}{12}, 0, \frac{G}{12}, 0, \frac{G}{12}, 0, -\frac{G}{3}, 0, -\frac{G}{3}, 0, -\frac{G}{3}, 0, -\frac{G}{3}, \right\rfloor_e$$

Wektor obciążenia zastępczego od sił powierzchniowych

wektor obciążenia zastępczego od sił powierzchniowych:

$\lfloor p \rfloor = \lfloor p_x, p_y \rfloor$



$$\underset{1\times 16}{\lfloor F^p \rfloor_e} = \int_{\Gamma_{pe}} \underset{1\times 2}{\lfloor p \rfloor}\underset{2\times 16}{[N]} d\Gamma_{pe} = t_e \int_0^l \lfloor p \rfloor [N] ds$$

$$= t_e \int_0^l \underset{1\times 2}{\lfloor p \rfloor}\underset{2\times 16}{[N]} ds = t_e \int_{-1}^{1} \underset{1\times 2}{\lfloor p \rfloor}\underset{2\times 16}{[N]} \frac{ds}{d\xi} d\xi$$

$$\frac{ds^2}{d\xi^2} = \frac{dx^2}{d\xi^2} + \frac{dy^2}{d\xi^2} \quad \rightarrow \frac{ds}{d\xi} = \sqrt{\left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2}$$

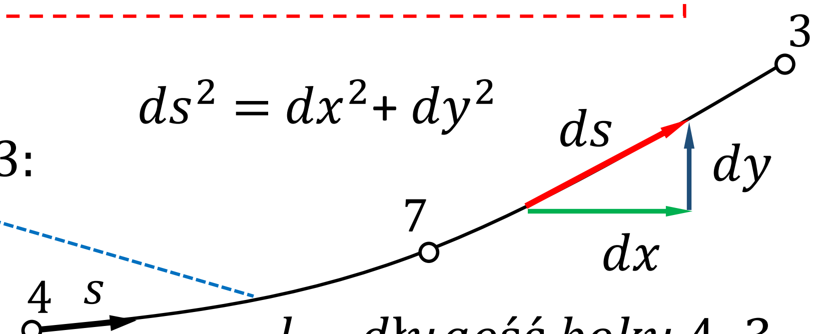
$$\boxed{\underset{1\times 16}{\lfloor F^p \rfloor_e} = t_e \int_{-1}^{1} \lfloor p_x, p_y \rfloor \underset{2\times 16}{[N]} \sqrt{\left(\underset{1\times 8}{\frac{\partial \lfloor N(\xi,1) \rfloor}{\partial \xi}} \underset{8\times 1}{\{x_i\}_e}\right)^2 + \left(\underset{1\times 8}{\frac{\partial \lfloor N(\xi,1) \rfloor}{\partial \xi}} \underset{8\times 1}{\{y_i\}_e}\right)^2} d\xi}$$

(wyliczamy numerycznie)

$ds^2 = dx^2 + dy^2$

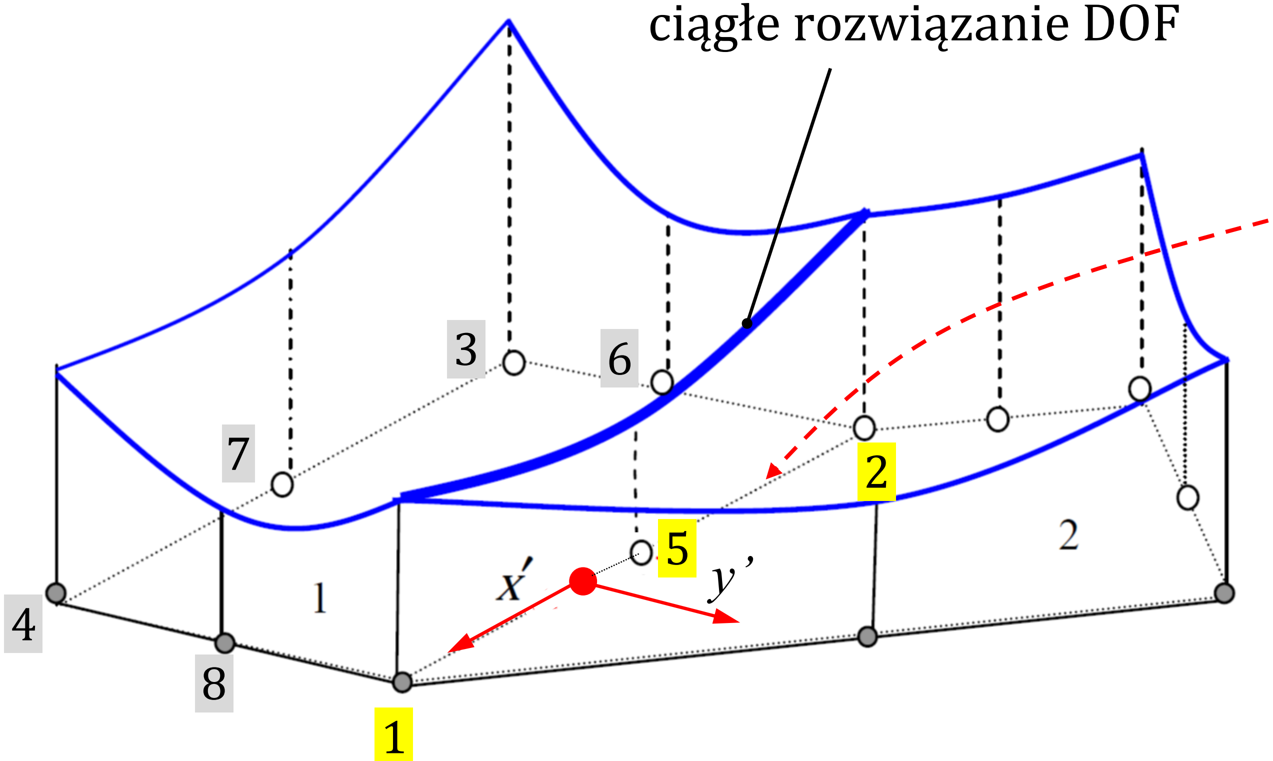
na boku 4–3:

$\eta = 1$



l – długość boku 4–3

Wyniki na granicy dwóch elementów 8-węzłowych



$$x = (x_1, x_5, x_2)$$
$$y = (y_1, y_5, y_2)$$

$$u = (u_1, u_5, u_2)$$
$$v = (v_1, v_5, v_2)$$

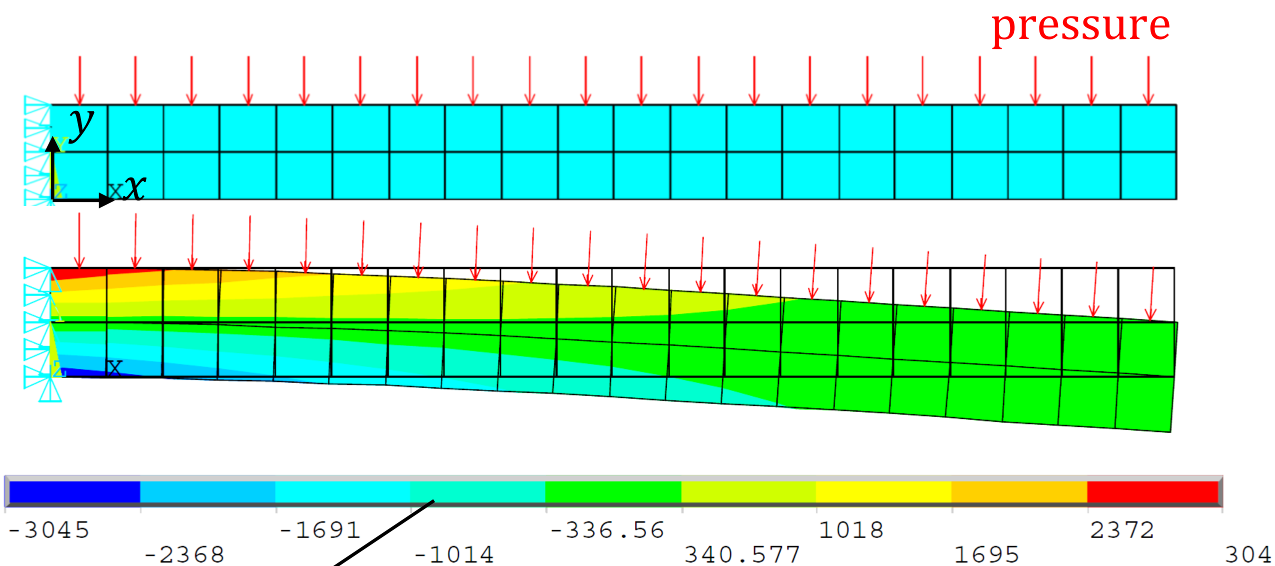
$$\left.\frac{\partial u}{\partial x'}\right|_1 = \left.\frac{\partial u}{\partial x'}\right|_2 \qquad \left.\frac{\partial u}{\partial y'}\right|_1 \neq \left.\frac{\partial u}{\partial y'}\right|_2 \Rightarrow \begin{array}{c}(\varepsilon_{x'})_1 = (\varepsilon_{x'})_2 \\ (\varepsilon_{y'})_1 \neq (\varepsilon_{y'})_2\end{array} \Rightarrow (\sigma_{ij})_1 \neq (\sigma_{ij})_2$$

nieciągłe rozwiązania elementowe

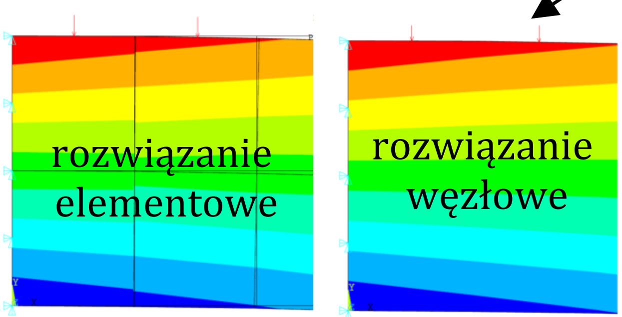
Przykład. 2D model belki wspornikowej *(8-węzłowe elementy)*

utwierdzenie

rozwiązanie elementowe σ_x



naprężenia σ_x



naprężenia σ_y